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Inverse formulation and finite difference solution for flow from a circular orifice

By ROLAND W. JEPPSON
Utah Water Research Laboratory,
College of Engineering, Utah State University

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The problem of flow from a large reservoir through a circular orifice is formulated by considering the velocity potential and Stokes's stream function as the independent variables and the radial and axial dimensions as the dependent variables, and a finite difference solution is obtained to the resulting boundary-value problem. This inverse formulation has the advantage over a finite difference solution in the physical plane that the region of flow is rectangular and consequently well adapted for minimum logic in programming a digital computer. The inverse finite difference solution is more readily obtained than a comparable solution in the physical plane, even though the inverse partial differential equation and associated boundary conditions are non-linear. The results from the inverse finite difference solution are in close agreement with other most recent results from approximate solutions to this problem.

The inverse method of solution is applicable to other free streamline as well as confined axisymmetric potential flow problems. The essential difference in other problems will be in the boundary conditions.†

1. Introduction

A class of problems of practical importance are those involving jets and cavities. In this class of problems viscous forces are generally confined to small regions of flow, and consequently are of minor importance, so that potential theory gives results adequate for most applications. Because of the close agreement between theoretical results and experimental measurements this class of potential free streamline problems has demonstrated the practical value of potential flow theory.

Theoretical hydraulicians, however, have often been frustrated because the available analytic methods generally require that the fluid be assumed weightless (i.e. the acceleration of gravity is zero), and are restricted to problems of plane flow. In fact exact solutions to axially symmetric potential flows with free surfaces have proved to be so formidable that researchers have been forced to obtain approximate solutions by numerical techniques, or other approximate methods even though such problems can be described by a two-dimensional co-ordinate system.

† *Keywords:* Orifice, Finite Differences, Non-linear Partial Differential Equation, Potential Flow.

The method described herein has been applied to the classical problem of axisymmetric, steady, potential flow from an infinite tank through a circular orifice. The approach is well adapted for obtaining approximate solutions to a variety of free streamline and confined axisymmetric potential flows. This particular problem has been selected because a considerable amount of work has been devoted to obtaining important parameters of the flow, such as the contraction coefficient, against which the results of the inverse finite difference solution can be compared. Furthermore, the nature of this problem will cause larger accumulative errors than those in many other axisymmetric problems. Consequently similar solutions of problems with short free streamlines, and for which the formulation does not involve approximating assumptions, would be expected to give accuracies at least as good as those obtained in the solution to this problem.

The first known approximate solution to the problem of flow from a reservoir through a circular orifice was given by Trefftz (1916): he formulated the problem in terms of a Fredholm integral equation and determined the position of the free streamline by trial and error. Trefftz's results indicated that the contraction coefficient is 0·61, very close to that for the plane slot with a contraction coefficient of 0·611. Later Southwell & Vaisey (1948) and also Rouse & Abul-Fetouch (1950) verified this coefficient with solutions by the relaxation method in the physical plane. For many the problem was considered adequately solved until Garabedian (1956) obtained an approximate solution by an ingenious dimensional perturbation scheme, which indicated the contraction coefficient equals 0·58. Hunt (1968) numerically solved the integral equation resulting from the surface distribution of vorticity, and substantiated Garabedian's result (that the contraction coefficient for axisymmetric potential flow from a reservoir through an orifice equals 0·58, somewhat smaller than the equivalent coefficient for the plane slot).

In this study the problem of potential flow from a reservoir through a circular orifice is formulated in the plane defined by the velocity potential function ϕ, and Stokes stream function ψ, for the radial and axial co-ordinates r and z. The solution to the resulting non-linear partial differential equation for $r(\phi, \psi)$ with associated boundary conditions is obtained by finite difference methods, and $z(\phi, \psi)$ subsequently obtained from the solution $r(\phi, \psi)$ by numerical differentiation and integration. A major advantage of this approach over finite difference (or relaxation) solutions in the physical plane is that the free surface boundary becomes straight with its position defined by the total discharge. The same approach has been successfully applied to axisymmetric flows through porous media (see Jeppson 1968). While the writer is not aware of others having used the general inverse approach to problems with axial symmetry, several investigators have used this or a similar formulation for two-dimensional potential flow problems (see Thom & Apelt 1961; Stanitz 1953; Markland 1965; Cassidy 1965 and Jeppson 1969).

2. Formulation

The partial differential equations which apply to r and z in the $\phi\psi$ plane can be obtained from implicit function theory. If $\phi = F(r,z)$ and $\psi = G(r,z)$, then r and z must be functions of ϕ and ψ, such that

$$\left.\begin{aligned} \frac{\partial r}{\partial \phi} &= \frac{1}{J}\frac{\partial \psi}{\partial z}, \quad \frac{\partial r}{\partial \psi} = -\frac{1}{J}\frac{\partial \phi}{\partial z}, \\ \frac{\partial z}{\partial \phi} &= -\frac{1}{J}\frac{\partial \psi}{\partial r}, \quad \frac{\partial z}{\partial \psi} = \frac{1}{J}\frac{\partial \phi}{\partial r}, \end{aligned}\right\} \tag{1}$$

in which J is the Jacobian determinant

$$J = \begin{vmatrix} \dfrac{\partial \phi}{\partial r} & \dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial \psi}{\partial r} & \dfrac{\partial \psi}{\partial z} \end{vmatrix}.$$

The needed equations for ϕ and ψ are obtained from the condition of irrotationality and the continuity equation, giving the following equations for the velocity components in the radial and axial directions respectively:

$$v = -\frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \tag{2}$$

and

$$w = -\frac{\partial \phi}{\partial z} = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{3}$$

Substituting from (1) into (2) and (3) respectively gives

$$\frac{\partial z}{\partial \psi} = \frac{1}{r}\frac{\partial r}{\partial \phi}, \tag{4}$$

and

$$\frac{\partial r}{\partial \psi} = -\frac{1}{r}\frac{\partial z}{\partial \phi}. \tag{5}$$

Equations (4) and (5) can be integrated to obtain $z(\phi,\psi)$ from a solution $r(\phi,\psi)$. The equations are

$$z = \int_{\phi} \frac{1}{r}\frac{\partial r}{\partial \phi}\, d\psi, \tag{6}$$

and

$$z = -\int_{\psi} r \frac{\partial r}{\partial \psi}\, d\phi, \tag{7}$$

in which the subscripts of the integral sign denote ϕ and ψ equal constant lines respectively.

Differentiating (4) with respect to ϕ and (5) with respect to ψ, and combining the results, leads to the following inverse equation for $r(\phi,\psi)$:

$$r^3\frac{\partial^2 r}{\partial \psi^2} + r\frac{\partial^2 r}{\partial \phi^2} + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 - \left(\frac{\partial r}{\partial \phi}\right)^2 = 0. \tag{8}$$

An equation exclusively for $z(\phi,\psi)$ cannot be obtained.

Since (8) is an elliptic equation, the approach used in obtaining a solution is to establish boundary conditions on all boundaries enclosing the region of flow, and solve the boundary-value problem by finite difference methods. Subsequently to obtaining the solution for $r(\phi, \psi)$ at each grid point of the finite difference mesh, (6) and (7) are solved numerically to obtain $z(\phi, \psi)$ at the same mesh points. The solution in this form is well adapted for plotting the flownet, since the radial and axial co-ordinates are given for each intersection of streamlines with equi-potential lines. Furthermore, other items of interest such as local velocity, pressure, velocity gradients and pressure gradients are readily obtained.

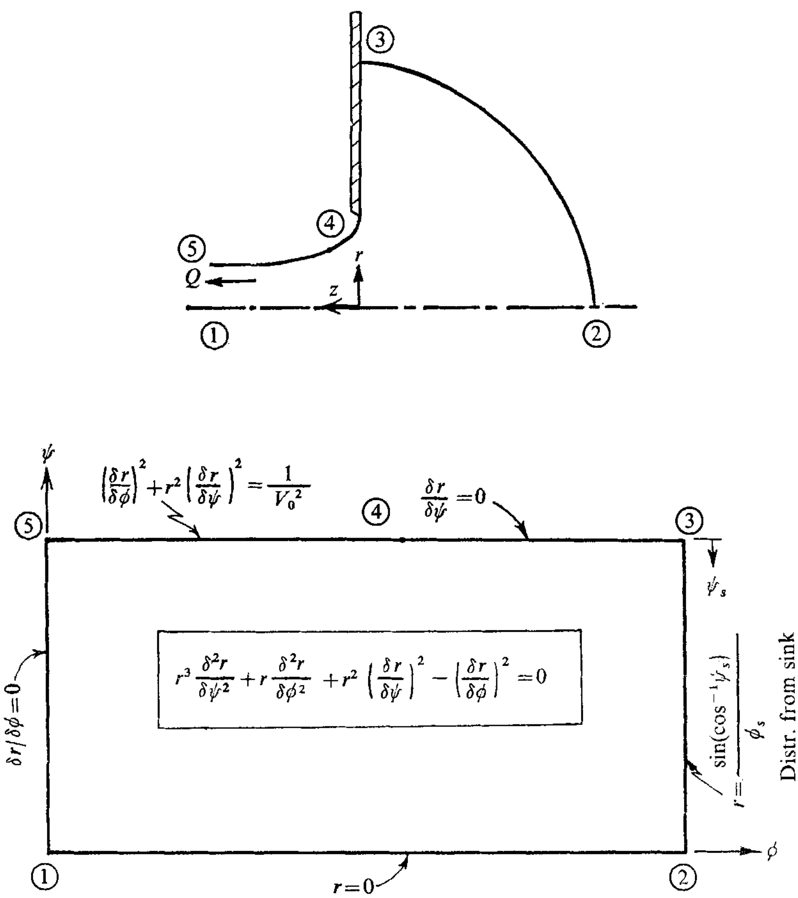

FIGURE 1. Formulation of axially symmetric flow of an ideal, weightless fluid from a large reservoir through a circular orifice.

The formulation of the boundary-value problem for $r(\phi, \psi)$ which describes flow from a large reservoir through a circular orifice is shown in figure 1. The boundary conditions for each boundary of the region are given in figure 1 by an equation near that boundary. In this formulation the assumption is made that a gravitational field does not exist and, consequently, that the velocity along the free streamline (i.e. boundary ④ to ⑤ on figure 1) is constant. For this boundary the condition is

$$\left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial r}{\partial \psi}\right)^2 = \frac{1}{V_0^2}, \tag{9}$$

in which V_0 (a constant which may be assigned a value of 1 without loss of generality) is the velocity along the free stream line. Equation (9) can be derived by expressing the right-hand sides of (1) and the Jacobian J in terms of the velocity and angle of the direction of flow, and then eliminating the angle by combining the equations.

It should be noted that this boundary condition can readily be altered to account for the presence of a gravitational field. If gravity is present, the discharge must be in the vertical direction to maintain axial symmetry, and the velocity in (9) is a function of the axial co-ordinate given by

$$V = (2g(H-z))^{\frac{1}{2}}, \tag{10}$$

in which H is the total fluid head.

Since the dependent variable z also occurs in the boundary condition when gravity is considered, a simultaneous solution of boundary-value problems for both $r(\phi, \psi)$ and $z(\phi, \psi)$ must be accomplished. While this case has not been investigated for this problem, it appears that the approach used for obtaining finite difference solutions to steady state seepage from ponds through porous media should be applicable (see Jeppson 1968).

The boundary condition within the reservoir denoted by ② to ③ in figure 1 is obtained by noting that a large distance within the reservoir the flow pattern will be very similar to that for a three-dimensional sink located at the orifice centre. The ϕ_s and ψ_s shown in the equation for this condition represent the potential and stream functions for a sink and as such are different from ϕ and ψ of the co-ordinate system. By selecting unity as the strength of this sink, ψ_s goes from a value of zero at ③ to 1 at ②. The value of ϕ_s is evaluated from the change in the potential function from ③ to ④ from

$$\phi_s = \frac{1}{r_4} - N_{34}/N_{23}, \tag{11}$$

in which r_4 is the radius of the orifice opening which may be taken equal to unity, N_{34} and N_{23} are the number of grid lines used in the finite difference solution, which is discussed later, between ③ to ④ and ② to ③ respectively. While this condition is only approximate, it can be made as good as desired by taking this boundary far enough within the reservoir and can be justified in light of the approxamite nature of the entire solution. All other boundary conditions are obvious.

It should be noted that the formulation in the $\phi\psi$ plane of the problem with a pipe preceding the orifice plate is a simple variation of that given in figure 1.

3. Finite differences

By finite difference methods the continuous dependent variable is replaced by discrete values at mesh points of a grid network imposed throughout the region of flow. The finite difference solution results upon solving the system of equations which 'link' the discrete values of the dependent variable at adjacent mesh points. The system of equations arises from applying a finite difference operator, which is developed from the differential equation, at all mesh points.

The finite difference operator for (8), which has been used, can be obtained by approximating the derivatives by second-order central differences. After some algebraic manipulation, and letting $\Delta\phi = \Delta\psi$, the finite difference operator can be written as a fourth-degree polynomial in terms of the value of r at the mesh point in question:

$$\left.\begin{aligned} f[r(I,J)] = [r(I,J)]^4 - \frac{r(I,J+1)+r(I,J-1)}{2}[r(I,J)]^3 \\ +\left[1-\frac{[r(I,J+1)-r(I,J-1)]^2}{8}\right][r(I,J)]^2 \\ -\frac{r(I+1,J)+r(I-1,J)}{2}r(I,J)+\frac{[r(I+1,J)-r(I-1,J)]^2}{8} = 0. \end{aligned}\right\} \quad (12)$$

In (12) the index I increases with ϕ, and J with ψ, such that $\phi = (I-1)\Delta\phi$ and $\psi = (J-1)\Delta\psi$.

The non-linearities in (8) cause the finite difference operator to be implicit, i.e. (12) cannot be solved explicitly for the value at the mesh point $r(I,J)$ under the assumption that the values at the surrounding mesh points are known. Therefore, methods such as the Gauss–Seidel iterative method with an over-relaxation factor (successive over-relaxation, see Forsythe & Wasow 1960), must be modified. In obtaining the solution to the system of equations denoted by (12), the successive over-relaxation method has been modified by what will be referred to as a 'Newton–Raphson inner iteration' in order to obtain the value of $r(I,J)$ at each mesh point which satisfies (12). The term 'inner iteration' distinguishes it from the Gauss–Seidel outer iteration, which sweeps across the mesh points adjusting the values to satisfy the finite difference operator continuously, until the change in the field values between consecutive sweeps (iterations) is less than some prescribed error. The method of solution can be described by the two iterative equations.

$$r^{(k+1)}(I,J) = r^{(k)}(I,J) - \omega[r_n(I,J) - r^{(k)}(I,J)], \quad (13)$$

and

$$r_n^{(m+1)}(I,J) = r_n^{(m)}(I,J) - \frac{f[r_n(I,J)]}{df/dr_n}, \quad (14)$$

in which the superscripts k and m denote the iteration number, $r_n(I,J)$ is the value supplied through the Newton–Raphson iterative (14) (to obtain a root for $f[r_n(I,J)]$, the function for the finite difference operator (12)), and ω is the over-relaxation factor (a value $\omega = 1{\cdot}55$ was used to obtain the solution presented later). The Gauss–Seidel iteration is given by (13).

For those boundaries for which values of r are not fixed, i.e. ③ to ④, ④ to ⑤ and ① to ⑤ on figure 1, a finite difference operator must be developed so these values become part of the unknowns in the system of equations for which a solution is sought. For the normal derivative conditions on boundaries ③ to ④ and ① to ⑤ the operator is (12), with the value of r at the image grid point just outside the boundary set equal to the value of r at the mesh point just inside the boundary.

The finite difference operator for the boundary condition of (9) (boundary ④ to ⑤) is obtained by approximating the derivatives by second-order central differences and combining with (12) to eliminate the value at the grid point outside the region of flow. The operator is

$$[r(I,N)]^4 - r(I,N-1)\,[r(I,N)]^3 + \left\{1 - \frac{1}{2}\left(\frac{4a^2}{V_0^2} - [r(I+1,N) - r(I-1,N)]^2\right)^{\frac{1}{2}}\right\}$$
$$\times [r(I,N)]^2 - \frac{r(I+1,N) - r(I-1,N)}{2}\, r(I,N)$$
$$+ \frac{1}{8}\left\{2[r(I+1,N) - r(I-1,N)]^2 - \frac{4a^2}{V_0^2}\right\} = 0, \quad (15)$$

in which $a = \Delta\phi = \Delta\psi$ is the spacing of the finite difference mesh, and N denotes the J index corresponding to the free streamline boundary.

In the application of (15) difficulty frequently occurs during the process of obtaining the finite difference solution because the argument of the square root becomes negative. When this occurs the expedient has been used to set the argument equal to zero, solve (15) for $r(I,N)$ and continue the iteration in hopes that the difficulty will disappear as the field becomes settled. This expedient has been successful for the problem of flow from a circular orifice.

Since the discharge is part of the solution and the specification of the free streamline velocity, the value of the mesh spacing cannot be specified. The value of the mesh spacing a in (15) has been determined after each outer iteration by computing the discharge from the orifice by multiplying the specified velocity V_0 by the area of the jet at section ① to ⑤ on figure 1. Since the radius at this section depends upon the cumulative effect of applying (15) along the free streamline, convergence to the correct radius is very slow. Experience gained from several tentative solutions obtained during the process of debugging the computer program indicates that convergence does occur fast enough to be practical if a coarse enough mesh is used. The approach, therefore, has been to use a coarser mesh to establish the final radius at point ⑤ on figure 1, before progressively obtaining the solutions for finer meshes.

A drawback of the method is that multiple roots satisfy the finite difference operator. To ensure that the Newton–Raphson iteration always selects the proper root requires that the initialization of the field values be reasonably good. It is difficult to define how good the initialization must be to ensure convergence because it no doubt depends upon the problem, the over-relaxation factor and other factors. An initialization of zeros, for example, would not result in a solution.

4. Results and discussion

The portion of the flownet beyond the circular orifice resulting from the solution is shown in figure 2. As mentioned earlier, since the solution consists of values for r and z at the intersection of each streamline and equi-potential line, a flownet is readily obtained by connecting consecutive co-ordinates with smooth curves. The flownet on figure 2 was drawn by a Calcomp plotter which draws essentially straight lines between consecutively supplied co-ordinates with a resolution of 0·005 in. Particularly near the centre-line a more pleasing appearance would result by connecting consecutive co-ordinates with smooth curves. The solution required approximately 3 min of UNIVAC 1108 execution time.

The location of the free streamlines agrees reasonably well along the entire length with Hunt's results, as can be seen in table 1. The free streamline given by this solution contracts slightly more adjacent to the orifice than Hunt's

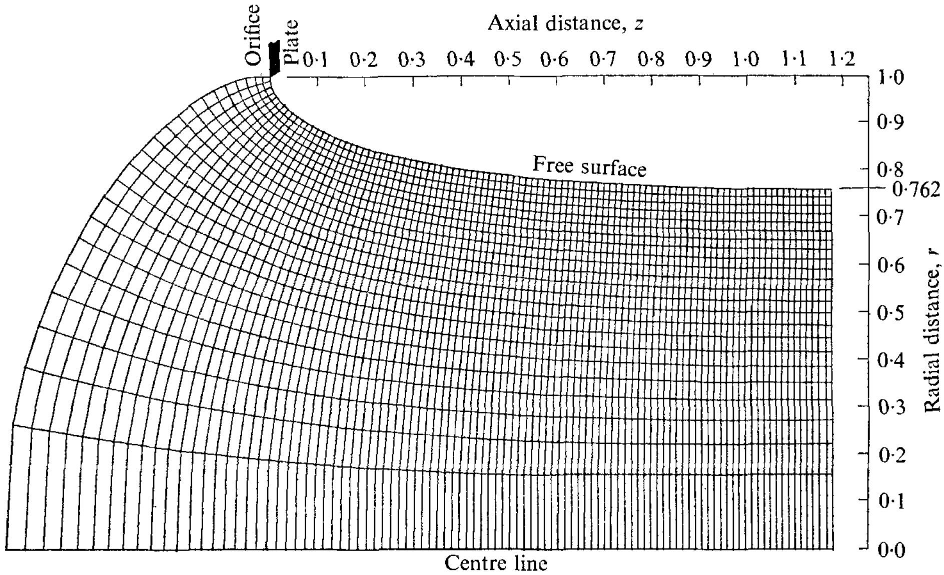

FIGURE 2. Flownet of jet issuing from orifice resulting from finite difference solution.

z/r_0	r/r_0 Hunt	r/r_0 Finite diff.
0·00	1·000	1·000
0·0125	0·971	0·962
0·0400	0·938	0·933
0·0813	0·905	0·902
0·1400	0·873	0·873
0·1975	0·850	0·851
0·2725	0·829	0·831
0·3575	0·813	0·813
0·4588	0·799	0·798
0·5700	0·788	0·786
0·6950	0·780	0·776
0·8188	0·774	0·769
0·9438	0·770	0·765
1·0713	0·767	0·763
1·2000	0·765	0·762

TABLE 1. Comparison of co-ordinates along free streamline

calculations. Also, the final radius given in table 1 is slightly less than Hunt's. This latter difference might well be the result of applying a condition for uniform flow across the section ① to ⑤ on figure 1, which is exact only at downstream infinity. Since an increase in the length of jet included in the region for which the solution is obtained causes a corresponding increase in the computational effort required, it is not practical to expand the region much beyond that given in figure 2.

A further observation made during debug runs on the computer is that the position of the free streamline is quite insensitive to how far the boundary ② to ③ on figure 1 is taken within the reservoir, provided it is several lengths of orifice opening within. The solution given in figure 2 placed this boundary approximately 10 lengths of orifice opening within the reservoir.

The solution is in an ideal form for determining other quantities of interest, such as local velocities, pressures or gradients throughout the flow field. Particularly, for the problem under investigation, the method has the disadvantage that the outcome of the solution depends heavily upon the cumulative errors resulting from applying the finite difference operator along boundary ④ to ⑤ in figure 1. Even with this dependency the accuracy achievable appears to be acceptable for many engineering applications. The disadvantage appears minor in light of the applicability of the approach to a wide variety of other axisymmetric potential flow problems, particularly since for other problems the accuracy of the solution will be less dependent on such cumulative effects.

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